

# Remark on Quantum Nambu Bracket

Chuan-Sheng Xiong

Department of Physics, Beijing University  
Beijing 100871, China  
email: xiong@ibm320h.phy.pku.edu.cn

## Abstract

We give an explicit realization of quantum Nambu bracket via matrix of multi-index, which reduces in the continuum limit to the classical Nambu bracket.

## 1 Introduction

In the original paper [1] Nambu introduced his famous Poisson bracket, which describes a generalized Hamilton mechanics. Let  $(x^i, i = 1, 2, \dots, n)$  be a set of dynamical variables which span a  $n$ -dimensional phase space. A Nambu-Poisson structure is defined as

$$\{f_1, f_2, \dots, f_n\} := \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} \frac{\partial f_1}{\partial x_{i_1}} \frac{\partial f_2}{\partial x_{i_2}} \dots \frac{\partial f_n}{\partial x_{i_n}}, \quad (1)$$

where  $\epsilon_{i_1, i_2, \dots, i_n}$  is the  $n$ -dimensional Levi-Civita tensor. This bracket has the following properties [2]

1. *Skew-symmetry*:

$$\{f_1, f_2, \dots, f_n\} = (-1)^{\epsilon(p)} \{f_{p(1)}, f_{p(2)}, \dots, f_{p(n)}\},$$

where  $p(i)$  is the permutation of indices and  $\epsilon(p)$  is the parity of the permutation.

2. *Derivation*:

$$\{f_1 f_2, f_3, \dots, f_{n+1}\} = f_1 \{f_2, f_3, \dots, f_{n+1}\} + \{f_1, f_3, \dots, f_{n+1}\} f_2.$$

3. *Fundamental Identity* (FI):

$$\begin{aligned} & \{\{f_1, f_2, \dots, f_n\}, f_{n+1}, \dots, f_{2n-1}\} = \{\{f_1, f_{n+1}, \dots, f_{2n-1}\}, f_2, \dots, f_n\} \\ & + \{f_1, \{f_2, f_{n+1}, \dots, f_{2n-1}\}, f_3, \dots, f_n\} + \dots + \{f_1, \dots, f_{n-1}, \{f_n, f_{n+1}, \dots, f_{2n-1}\}\}, \end{aligned}$$

which is a generalization of the Jacobi identity.

The dynamics of a Nambu system is determined by  $(n - 1)$  Hamiltonians  $H_1, H_2, \dots, H_{n-1}$  and is described by Nambu-Hamiltonian equation

$$\frac{df}{dt} = \{f, H_1, H_2, \dots, H_{n-1}\}. \quad (2)$$

The quantization of Nambu bracket turns out to be a quite non-trivial problem. In general one expects that the quantum Nambu bracket satisfies some properties analogous to those of the classical Nambu bracket, i.e.

1. *Skew-symmetry*,

$$[A_1, A_2, \dots, A_n] = (-1)^{\epsilon(p)} [A_{p(1)}, A_{p(2)}, \dots, A_{p(n)}]. \quad (3)$$

where  $p(i)$  is the permutation of indices and  $\epsilon(p)$  is the parity of the permutation.

2. *Derivation*:

$$[A_1 A_2, A_3, \dots, A_{n+1}] = A_1 [A_2, A_3, \dots, A_{n+1}] + [A_1, A_3, \dots, A_{n+1}] A_2. \quad (4)$$

3. *Fundamental Identity* (FI):

$$\begin{aligned} & [[A_1, A_2, \dots, A_n], A_{n+1}, \dots, A_{2n-1}] = [[A_1, A_{n+1}, \dots, A_{2n-1}], A_2, \dots, A_n] \\ & + [A_1, [A_2, A_{n+1}, \dots, A_{2n-1}], A_3, \dots, A_n] + \dots + [A_1, \dots, A_{n-1}, [A_n, A_{n+1}, \dots, A_{2n-1}]]. \end{aligned} \quad (5)$$

To maintain the skew-symmetry, it is natural to introduce the following quantum commutator[1]

$$[A_1, A_2, \dots, A_n] := \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} (A_{i_1} A_{i_2} \dots A_{i_n}). \quad (6)$$

Thus the problem of quantizing Nambu bracket reduces to finding a proper quantum product, which maps  $n$  objects to one, such that the quantum commutator (6) satisfies the derivation law (4) and the fundamental identity (5). This turns out to be a very difficult problem. Several quantization schemes have been proposed in the literatures[3][4][5]. Among them, the Zariski quantization is a deformation quantization, which obeys all the three requirements listed in the above, and has a clear relationship with the classical Nambu bracket [3]. In the other approaches, the quantum Nambu bracket does not satisfy the derivation law [4][5]. In particular, Awata etc. proposed a matrix model realization of the quantum Nambu bracket[5], but its link to the classical Nambu bracket is quite obscure. We will give another matrix realization of the quantum Nambu bracket, which reduces in the continuum limit to the classical Nambu bracket.

## 2 Quantum Nambu bracket

Let each of  $\{A_i, i = 1, 2, \dots, n\}$  denote a  $\overbrace{N \times N \times \dots \times N}^n$  matrix, and  $\mathcal{S}_n(N)$  the collection of all such matrices. we define the quantum product as follows \*

$$(A^{(1)}A^{(2)} \dots A^{(n)})_{i_1 i_2, \dots, i_n} := \sum_l A_{i_1 i_2 \dots i_{n-1} l}^{(1)} \dots A_{i_1 l i_3 \dots i_n}^{(n-1)} A_{l i_2 \dots i_n}^{(n)}. \quad (7)$$

We may also define the inner scalar product, i.e. the “Trace”

$$\text{Tr}(A) := \sum_{i=1}^N A_{ii \dots i}. \quad (8)$$

When  $n = 2$ , we recover the usual product of the square matrix. When  $n \geq 3$ , the quantum product (7) is *non-commutative* and *non-associative*. The derivation law (4) does not hold since the product of two objects  $A_1$  and  $A_2$  does not make sense. Even worse, the above quantum product generally does not make the quantum commutator (6) satisfying the Fundamental Identity (5). We would expect that there exists some subset of  $\mathcal{S}_n(N)$ , which is closed under the quantum commutator (6), and obeys on the Fundamental Identity (5). We will give some examples to show that this is indeed the case.

In this article we only consider the quantization of the the even-dimensional Nambu bracket. To do so, set  $n = 2m(m \geq 2)$ . Let  $\mathcal{AS}_{2m}(N)$  denote the subset of  $\mathcal{S}_{2m}(N)$  containing all matrices which are totally antisymmetric, i.e.

$$A_{i_1 i_2 \dots i_{2m}} = (-1)^{\epsilon(p)} A_{p(i_1) p(i_2) \dots p(i_{2m})}.$$

It is obvious to see that this truncation is consistent with the quantum commutator(6), if we use the quantum product (7). When  $N < 2m$ , the subset  $\mathcal{AS}_n(N)$  is empty. When  $N = 2m$ ,  $\mathcal{AS}_n(N)$  contains just one element, which generates an abelian algebra. So the Fundamental Identity is trivial. The first non-trivial example is the case  $N = 2m + 1$ , where  $\mathcal{AS}_{2m}(N)$  contains  $N$  independent elements, which we may denote by

$$(T_a)_{i_1 i_2 \dots i_{2m}} := \epsilon_{a i_1 i_2 \dots i_{2m}}, \quad a = 1, 2, \dots, N. \quad (9)$$

It is straightforward to show that these matrices generate a quantum algebra with the following commutation relation

$$[T_{a_1}, T_{a_2}, \dots, T_{a_{2m}}] = \sum_b \epsilon_{a_1 a_2 \dots a_{2m} b} T_b. \quad (10)$$

The fundamental identity is guaranteed by the following equality

$$\sum_c \left( \epsilon_{c a_1 a_2 \dots a_n} \epsilon_{c b_1 b_2 \dots b_n} - \sum_{i=1}^n (a_i \leftrightarrow b_n) \right) = 0. \quad (11)$$

One may generalize the consideration here to the case with the general value of  $N$ .

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\*Throughout the paper, any summation index will be specified explicitly.

### 3 The continuum limit

The quantum product (7) has a very simple geometric meaning. Consider  $N(N \geq n)$  ordered points in  $(n-1)$ -dimensional Euclidean space  $\mathbf{R}^{n-1}$ . For each  $(n-1)$ -dimensional convex with vertices  $(i_1, i_2, \dots, i_n)$  in general positions, we associate to it a matrix element  $A_{i_1, i_2, \dots, i_n}$ . Adding a point  $l$  to this convex leads to  $n$  more convexes, all of which contain the point  $l$ . Thus we have a natural composition law,

$$\underbrace{\mathcal{S} \times \mathcal{S} \times \dots \times \mathcal{S}}_n \longrightarrow \mathcal{S}, \quad (A_{i_1, i_2, \dots, i_n}) \in \mathcal{S}, \quad (12)$$

which glues  $n$  convexes to one, and defines a general “group structure”. The explicit form of this composition law is given by the quantum product (7).

The geometric picture provides a way to analyse the continuum limit of the quantum Nambu bracket. To do so, we may take the following viewpoint. Each matrix defines a map from the set of convexes to the field  $\mathbf{R}$  or  $\mathbf{C}$ ,

$$A : \{\text{convexes}\} \longrightarrow \mathbf{R}(\text{or } \mathbf{C}).$$

Each convex can be characterized by its center and volume <sup>†</sup>, which is

$$\begin{aligned} \vec{x} &:= \frac{\epsilon}{n}(\vec{i}_1 + \vec{i}_2 + \dots + \vec{i}_n) = (x_1, x_2, \dots, x_{n-1}), \quad \epsilon := \frac{1}{N}, \\ \text{vol} &:= \frac{1}{(n-1)!} \epsilon_{\mu_1 \mu_2 \dots \mu_{n-1}} (\vec{i}_1 - \vec{i}_n)_{\mu_1} (\vec{i}_2 - \vec{i}_n)_{\mu_2} \dots (\vec{i}_{n-1} - \vec{i}_n)_{\mu_{n-1}}, \\ \vec{v}^{(j)} &= \vec{i}_{n-j+1} - \vec{i}_n, \quad 1 \leq j \leq n. \end{aligned}$$

Where  $(\mu_1, \mu_2, \dots, \mu_{n-1})$  are indices of the Cartesian coordinates in the  $(n-1)$ -dimensional space. To go to the continuum limit, we let all the  $n$ -indices  $(i_1, i_2, \dots, i_n)$  go to infinity, but their differences finite. Thus  $\vec{x}$  become the continuum variables. To characterize the dependence of a matrix on the volume of the convex, we introduce a new parameter  $x_n$ , such that

$$\frac{\partial f(\vec{x}, x_n)}{\partial x_n} := f(\vec{x}, x_n) \cdot \text{vol},$$

here  $f(\vec{x}, x_n)$  is the continuum limit of the matrix  $A_{i_1, i_2, \dots, i_n}$ . One may think of  $x_n$  as the constant density of mass, momentum or charge, etc. With these notations, we may expand the commutator (6) in the powers of  $\epsilon$ , and we find that

$$[A^{(1)}, A^{(2)}, \dots, A^{(n)}] = \left( \frac{n! \epsilon^{n-1}}{n^{n-1}} \right) \{f^{(1)}, f^{(2)}, \dots, f^{(n)}\} + \mathcal{O}(\epsilon^n). \quad (13)$$

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<sup>†</sup>We may think of that the ordered points are sited on a  $(n-1)$ -dimensional rectangular lattice. The volume-preserving symmetry group of the lattice is  $sl(n-1, \mathbf{Z})$ . We are interested in the quotient lattice space, which is the regular lattice modulo out  $sl(n-1, \mathbf{Z})$  symmetry.

## 4 Conclusion

In this letter we have constructed a quantum Nambu bracket via matrices of multi-index. We have shown that our quantum commutator has the right continuum limit. Our construction may be helpful to quantize  $p$ -brane theory, just like what we did in studying 2-dimensional quantum gravity by the ordinary matrix models. Furthermore if we regard the points in  $\mathbf{R}^{n-1}$  as a set of  $D0$ -particles, we may expect a matrix theory in more general setting. We wish to discuss these problems in more details elsewhere.

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